Exceptional points in open quantum systems

2008 J. Phys. A: Math. Theor. 41 244018

(http://iopscience.iop.org/1751-8121/41/24/244018)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 244018 (15pp) doi:10.1088/1751-8113/41/24/244018

Exceptional points in open quantum systems

Markus Müller[1,2] **and Ingrid Rotter**[2]

[1] Facultad de Ciencias, Universidad Autónoma del Estado de Morelos, 62210 Cuernavaca, Morelos, Mexico
[2] Max-Planck-Institut für Physik komplexer Systeme, D-01187 Dresden, Germany

E-mail: rotter@mpipks-dresden.mpg.de

Received 12 February 2008
Published 3 June 2008
Online at stacks.iop.org/JPhysA/41/244018

Abstract

Open quantum systems are embedded in the continuum of scattering wavefunctions and are naturally described by non-Hermitian Hamilton operators. In the complex energy plane, exceptional points appear at which two (or more) eigenvalues of the Hamilton operator coalesce. Although they are a countable set of single points in the complex energy plane and therefore of measure zero, they determine decisively the dynamics of open quantum systems. A powerful method for the description of open quantum systems is the Feshbach projection operator formalism. It is used in the present paper as a basic tool for the study of exceptional points and of the role they play for the dynamics of open quantum systems. Among others, the topological structure of the exceptional points, the rigidity of the phases of the eigenfunctions in their vicinity, the enhancement of observable values due to the reduced phase rigidity and the appearance of phase transitions are considered. The results are compared with existing experimental data on microwave cavities. In the last section, some questions being still unsolved, are considered.

PACS numbers: 03.65.Vf, 03.65.Ta, 03.65.Yz, 05.60.Gg

1. Introduction

The existence of exceptional points has been discussed in mathematical literature for more than 40 years, see e.g. the textbook [1]. At these points, k eigenvalues of a non-Hermitian operator coalesce, and the corresponding eigenvectors are linearly dependent. Simultaneously, $k-1$ associated vectors appear which form, together with the single eigenvector, a Jordan block. These k vectors span the so-called k-dimensional algebraic eigenspace so that the total space dimension is preserved at the exceptional point [1, 2].

The Hamilton operator that describes effectively the properties of an open quantum system is non-Hermitian. Its complex eigenvalues provide not only the positions of the

resonance states but also their (finite) lifetimes. The general formalism for the description of open quantum systems has been formulated by Feshbach [3] already more than 40 years ago. Nevertheless, spectroscopic studies on concrete nuclei have been performed, in the past, almost exclusively either by using statistical approximations (at high-level density) or on the basis of the R matrix theory (when the resonance states are well separated). Spectroscopic information on the positions and lifetimes of narrow resonance states is obtained usually by continuing the S matrix into the complex energy plane and considering its poles.

In past years, high-resolution experiments became possible in different fields of physics. The obtained results turned out to be a challenge for the theory. They call for a theoretical description which is more precise than the standard one. Particularly in the regime of overlapping resonances, the non-Hermiticity of the Hamilton operator has to be considered. Also the influence of exceptional points should be studied in detail.

In the physical literature, the notation *exceptional point* is seldom used. Instead, other notations can be found. Mostly, the exceptional points are identified with *double poles* of the S matrix (although this identification holds only approximately, since the poles of the S matrix are just an auxiliary means for the extraction of spectroscopic information used in standard theory with Hermitian Hamilton operator, see [4]). Double poles of the S matrix are considered, quite generally, in scattering theory [5]. They are studied in some detail in, e.g., [6, 7]. Numerical calculations for atoms in a laser field are provided in [8, 9]. In other studies on atoms, the notation *hidden crossings* is used [10]. Due to their relation to the avoided level crossing phenomenon in the continuum, the exceptional points are called often *crossing points* in the continuum meaning that the eigenvalue trajectories, traced as a function of a parameter, cross at these points [11]. The parameter values at which the crossing occurs are called *critical* values. The physical meaning of the crossing points arises from the fact that they are *branch points* in the continuum at which level repulsion (and small changes of the widths) passes into widths bifurcation (accompanied by level clustering) and vice versa [11]. The exceptional points determine the spectroscopic properties of realistic open quantum systems in the regime of overlapping resonances, i.e. of almost all open quantum systems under realistic conditions. This statement has been confirmed by numerical studies on atoms [8, 9] and quantum dots, e.g. [12–15]. The notation *exceptional point* has been used now also in the study of atomic spectra [16].

The significance of exceptional points for physical processes expresses itself by their topological structure. It is considered in [17, 18] and studied experimentally on a quantum billiard [19, 20]. The geometric phase appearing by encircling an exceptional point differs from the Berry phase. It does not pass into the Berry phase even in the limit of vanishing coupling to the continuum [13].

Another interesting question is the structure of an exceptional point with three or more coalescent eigenvalues. The limiting case $N \to \infty$ is considered about 10 years ago [17, 21]. In these papers, the relation between an N-fold exceptional point and a phase transition is discussed. The results for numerical calculations on a realistic system (quantum dot) where such a situation is almost realized, are provided in [15].

An N-fold exceptional point may occur when $N-1$ twofold exceptional points coalesce or when, for some symmetry involved in the system, some of the states remain unaffected by the eigenvalue crossing. The last situation is studied in an open quantum dot with three resonance states [13]. In this context, the experimental data obtained for the Berry phase in a closed quantum billiard are interesting where triple diabolic points with a certain internal symmetry are observed [22]. The problem of multiple exceptional points is also discussed during the 6th International Workshop on Pseudo Hermitian Hamiltonians in Quantum Physics [23–26].

In the following, the role of exceptional points in the dynamics of open quantum systems will be reviewed. Our considerations are based on the results of analytical as well as numerical studies by using both toy models and models describing realistic systems. Most calculations are performed by using the Feshbach projection operator (FPO) formalism. The aim of the review is to discuss the different results from a common point of view in order to receive a better understanding of the underlying physics and to formulate unsolved problems as clear as possible.

In section 2, the FPO formalism is sketched with special consideration of the phases of the wavefunctions of the resonance states. Geometric phases of diabolic and exceptional points are considered in section 3 while the relation of exceptional points to phase transitions is discussed in section 4. In section 5, some numerical examples are mentioned which show the influence of exceptional points in realistic systems. Some conclusions are drawn in the last section including reference to some yet unsolved problems.

2. Exceptional points in the framework of the Feshbach projection operator (FPO) formalism

2.1. Eigenvalues and eigenfunctions of the non-Hermitian Hamilton operator $H_{\rm eff}$

In the present paper, the FPO technique [3] will be used in order to describe the properties of open quantum systems. In this formalism, the spectroscopic properties follow from the complex eigenvalues and eigenfunctions of a non-Hermitian symmetrical Hamilton operator. The condition for the appearance of exceptional points can be formulated explicitly.

In the FPO formalism, the full function space is divided into two subspaces: the Q subspace contains all wavefunctions that are localized inside the system and vanish exponentially outside while the wavefunctions of the P subspace are extended up to infinity and vanish inside the system, see [11]. It is $P + Q = 1$. In this formalism, two Hamilton operators characterize the system. The first one, H, is Hermitian. It describes the scattering in the whole function space,

$$(H - E)\Psi_C^E = 0, \tag{1}$$

consisting of the two subspaces: the subspace of discrete states of the considered (closed) system (described by the Hermitian operator H_B) and of the subspace of scattering states (continuum described by the Hermitian operator H_C) into which the system is embedded. In solving (1) in the whole function space by using the FPO technique [3], the effective non-Hermitian Hamilton operator

$$H_{\rm eff} = H_B + \sum_C V_{BC} \frac{1}{E^+ - H_C} V_{CB} \tag{2}$$

appears which contains H_B as well as an additional symmetrical non-Hermitian term that describes the coupling of the discrete states via the common environment of scattering wavefunctions. Here V_{BC}, V_{CB} stand for the coupling matrix elements between the *eigenstates* of H_B and the environment [11] that may consist of different continua C. The operator $H_{\rm eff}$ is non-Hermitian,

$$(H_{\rm eff} - z_\lambda)\phi_\lambda = 0, \tag{3}$$

its eigenvalues z_λ and eigenfunctions ϕ_λ are complex. The eigenvalues provide not only the energies E_λ of the resonance states but also their widths Γ_λ (inverse lifetimes). The Hamilton operator $H_{\rm eff}$ describes the spectroscopic properties of the system localized in the Q subspace

and embedded in the P subspace. The eigenstates of $H_{\rm eff}$ have, in general, a finite lifetime due to their decay into the P subspace.

The eigenvalues and eigenfunctions of H_B contain the interaction u of the discrete states which is given by the nondiagonal matrix elements of H_B. This interaction is characteristic of the corresponding closed system and may be called therefore internal interaction. The eigenvalues and eigenfunctions of $H_{\rm eff}$ contain additionally the interaction v of the resonance states via the common continuum (v is used here instead of the concrete matrix elements of the second term of $H_{\rm eff}$). This part of interaction is, formally, of second order and may be called external interaction. While u and $\mathrm{Re}(v)$ cause level repulsion in energy and influence the widths only marginally, $\mathrm{Im}(v)$ is responsible for the bifurcation of the widths of the resonance states and level clustering in energy. The bifurcation of the widths causes the formation of a few short-lived resonance states together with long-lived narrow ones (resonance trapping) [11]. This phenomenon has been proven experimentally in a microwave cavity [27].

Since the effective Hamilton operator (2) depends explicitly on the energy E, $H_{\rm eff} = H_{\rm eff}(E)$, so do its eigenvalues z_λ and eigenfunctions ϕ_λ. Far from thresholds, the energy dependence is weak within an energy interval of the order of magnitude of the width of the resonance state. It should be underlined here that the FPO formalism allows us to describe the open quantum system also in the regime of strongly overlapping resonance states and that the S matrix is always unitary. The energy dependence of the coupling coefficients calculated by means of ϕ_λ, is a direct consequence of the unitarity condition, see e.g. [28].

The solutions of the fixed-point equations $E_\lambda = \mathrm{Re}(z_\lambda)_{|E=E_\lambda}$ and of $\Gamma_\lambda = -2\,\mathrm{Im}(z_\lambda)_{|E=E_\lambda}$ are numbers that coincide (approximately) with the poles of the S matrix. Using the FPO formalism with non-Hermitian Hamilton operator $H_{\rm eff}$, it is however not necessary to look for the poles of the S matrix since the spectroscopic information is directly given by the complex eigenvalues z_λ of $H_{\rm eff}$. In the S matrix, the eigenvalues z_λ with their full energy dependence appear. Due to this fact, the S matrix contains information on the environment of the considered resonance states such as the position of decay thresholds and resonance states overlapped by them.

Thus, the FPO formalism may be considered as an extension [29] of the R matrix theory used in standard quantum mechanics for the description of decaying states. The standard spectroscopic parameters (positions, widths and partial widths of the resonance states λ) are replaced by the energy-dependent functions E_λ, Γ_λ and coupling matrix elements between system and environment that are calculated by means of the eigenfunctions ϕ_λ of $H_{\rm eff}$. While R matrix theory gives reasonable results only for narrow non-overlapping resonance states, the FPO formalism can be used for all resonance states including the short-lived ones in the overlapping regime. The influence of neighboring resonances as well as of decay thresholds is taken into account via the energy dependence of the eigenvalues z_λ and eigenfunctions ϕ_λ. In many systems, the spectroscopic properties can be controlled by means of an external parameter. The results obtained in the FPO formalism pass into those of the R matrix theory in the regime of non-overlapping resonance states.

2.2. *Phase rigidity r_λ of the eigenfunctions ϕ_λ of the non-Hermitian Hamilton operator $H_{\rm eff}$*

The eigenfunctions ϕ_λ of the non-Hermitian symmetrical Hamilton operator $H_{\rm eff}$ are complex and biorthogonal, i.e. the left and right eigenvectors differ from one another. Due to the symmetry of $H_{\rm eff}$, it is $\left\langle \phi_\lambda^{\rm left}\right| = \left\langle \phi_\lambda^{\rm right*}\right| \equiv \left\langle \phi_\lambda^{*}\right|$. In contrast to $\langle\phi_\lambda|\phi_\lambda\rangle$, the value $\langle\phi_\lambda^*|\phi_\lambda\rangle$ is complex. Nevertheless, it can be used to normalize the biorthogonal wavefunctions [11, 30]. Choosing the orthonormality conditions as

$$\langle\phi_\lambda^*|\phi_{\lambda'}\rangle = \delta_{\lambda,\lambda'} \tag{4}$$

the transition is smooth from the wavefunctions of an open quantum system (with, in general, nonvanishing decay widths Γ_λ of its states and biorthogonal wavefunctions ϕ_λ) to those of the corresponding closed one (with $\Gamma_\lambda \to 0$ and real wavefunctions that are normalized in the standard manner). That means $\langle \phi_\lambda^* | \phi_\lambda \rangle \to \langle \phi_\lambda | \phi_\lambda \rangle = 1$ if the coupling vectors in the non-Hermitian part of (2) vanish. As a consequence [11, 31]

$$\langle \phi_\lambda | \phi_\lambda \rangle \equiv A_\lambda \geqslant 1 \tag{5}$$

$$B_\lambda^{\lambda'} \equiv \langle \phi_\lambda | \phi_{\lambda' \neq \lambda} \rangle = -B_{\lambda'}^{\lambda} \equiv -\langle \phi_{\lambda' \neq \lambda} | \phi_\lambda \rangle \quad \left| B_\lambda^{\lambda'} \right| \geqslant 0. \tag{6}$$

The normalization condition (4) entails that the phases of the eigenfunctions in the overlapping regime are not rigid: the normalization condition $\langle \phi_\lambda^* | \phi_\lambda \rangle = 1$ is fulfilled only when

$$\mathrm{Im}\, \langle \phi_\lambda^* | \phi_\lambda \rangle = 0. \tag{7}$$

Since ϕ_λ and, as a consequence, also the value $\mathrm{Im}\langle \phi_\lambda^* | \phi_\lambda \rangle$ depends on parameters, the condition (7) corresponds generally to a rotation of the eigenvector by a certain angle β_λ (phase change of the wavefunction by β_λ) when the parameters are varied.

Let us fix the phases of the wavefunctions of the original states corresponding to $v = 0$ (vanishing non-diagonal matrix elements of the second term of (2)) to $\beta_\lambda^0 = 0$ or $\pm\pi$, so that $\mathrm{Im}\phi_\lambda^0 = 0$. The influence of a neighboring state is described by $v \neq 0$ (i.e. by the non-diagonal matrix elements of the second term of (2)). At $v \neq 0$, the angle β_λ is different from β_λ^0, generally. The difference $\left| \beta_\lambda - \beta_\lambda^0 \right|$ may be $\pm\pi/4$ at most, corresponding to $\mathrm{Re}\phi_\lambda = \pm \mathrm{Im}\, \phi_\lambda$ (as compared to $\mathrm{Im}\phi_R^0 = 0$). This maximum value occurs at an exceptional point where two eigenvalues z_λ, $z_{\lambda'}$ of H_{eff} coalesce. Here [11, 12, 13, 30, 32]

$$\phi_\lambda \to \pm \mathrm{i} \phi_{\lambda'}; \qquad \phi_{\lambda'} \to \mp \mathrm{i} \phi_\lambda. \tag{8}$$

This relation between the two wavefunctions at the exceptional point has been found also in numerical studies on a realistic system (laser-induced continuum structures in atoms [9]).

The phase rigidity defined by

$$r_\lambda = \frac{\langle \phi_\lambda^* | \phi_\lambda \rangle}{\langle \phi_\lambda | \phi_\lambda \rangle} = \frac{1}{(\mathrm{Re}\, \phi_\lambda)^2 + (\mathrm{Im}\, \phi_\lambda)^2} = \frac{1}{A_\lambda} \tag{9}$$

is a useful measure [15, 33] for the rotation angle β_λ. When the resonance states are distant from one another, it is $r_\lambda \approx 1$ due to $\langle \phi_\lambda | \phi_\lambda \rangle \approx \langle \phi_\lambda^* | \phi_\lambda \rangle$. In approaching an exceptional point in the complex energy plane [11, 13], we have $\langle \phi_\lambda | \phi_\lambda \rangle \equiv A_\lambda \to \infty$ and $r_\lambda \to 0$. Therefore $1 \geqslant r_\lambda \geqslant 0$.

It should be underlined that, after defining the normalization condition (4), the values r_λ are fixed by the coupling matrix elements v of H_{eff} which determine the degree of overlapping of the resonance states. They can be varied by controlling the system by means of external parameters, e.g. by means of a laser in the case of an atom with many levels (for concrete examples see [8, 9]). The rotation angle β_λ as well as the values A_λ and r_λ may be considered to be a synonym for the biorthogonality of the eigenfunctions ϕ_λ of the non-Hermitian Hamiltonian (2). They are measures for the distance of the considered states from an exceptional point in the complex plane and for the spectroscopic reordering processes occurring in an open quantum system with overlapping resonance states under the influence of the coupling to the continuum. Physically, the phase rigidity r_λ measures the degree of alignment of one of the overlapping resonance states with one of the scattering states ξ_C^E of the environment. This alignment takes place at the cost of the other states that decouple, to a certain extent, from the environment (*widths bifurcation* or *resonance trapping* occurring in the neighborhood of an exceptional point [11]). For the same parameter set, the values r_λ are, generally, different for the different states λ.

2.3. Eigenfunctions of H_{eff} at and in the vicinity of an exceptional point

An exceptional point is defined by the coalescence of (at least) two eigenvalues: $z_\lambda^0 \equiv z_\lambda = z_{\lambda'}$ where $z_\lambda = E_\lambda - \mathrm{i}/2\Gamma_\lambda$. According to section 2.2, the mathematical consequences for the eigenfunctions $\phi_\lambda^0 \equiv \phi_\lambda$ at an exceptional point are the following.

(i) The eigenfunctions ϕ_λ^0 and $\phi_{\lambda'}^0$ are linearly dependent according (4) to (8) with $A_\lambda \to \infty$ and $\left|B_\lambda^{\lambda'}\right| \to \infty$,
(ii) The phase rigidity is zero, $r_\lambda = 0$ according to (8), i.e. $\mathrm{Re}(\phi_\lambda) = \mathrm{Im}(\phi_\lambda)$,
(iii) The phases of the eigenfunctions are ill defined due to the jump by $\pi/4$ at the exceptional point (see [13, 30] for details).

Furthermore, the eigenvectors ϕ_λ^0 and ϕ_λ^{*0} are supplemented by the corresponding associated vectors defined by Jordan chain relations, see equation (13) in [30]. The wavefunction at the exceptional point is chiral-like since it can be represented by $\phi_\lambda \pm \mathrm{i}a\phi_{\lambda'}$ with real a according to (8). Only for $a = 1$ (neglecting the phase jump), the wavefunction can be considered to be chiral.

Relations (i)–(iii) hold strictly only at the exceptional point, i.e. at a single point in the complex parameter plane. However, the exceptional points influence strongly the neighborhood where the resonance states overlap in energy due to their finite decay widths. Here,

(i) $A_\lambda > 1$ (but finite) and $\left|B_\lambda^{\lambda'}\right| \neq 0$,
(ii) the phase rigidity is reduced, $0 < r_\lambda < 1$,
(iii) the height of the phase jump is $\pi/4$ and takes the form of a Heaviside step function (see equation (49) in [30]).

Due to these relations, the exceptional points strongly influence the physical properties of open quantum systems in the regime of overlapping resonances, i.e. in a large parameter range. They separate the regime of level repulsion (and small influence on the widths) from that of widths bifurcation (accompanied by level clustering). For details see [11, 28, 32]. Numerical results for realistic systems are considered in section 5.

3. Geometric phases of diabolic and exceptional points

3.1. Geometric phase in the function space of discrete states (Berry phase)

Since the pioneering papers [34, 35], the Berry phase is understood to be inherent in quantum mechanics. It is directly related to the function space used in the description of closed quantum mechanical systems with discrete eigenvalues. The interaction u between the states is of standard first-order type.

For nonvanishing u, two eigenvalue trajectories of the (Hermitian) Hamilton operator do not cross. This follows from the eigenvalue equation for a Hermitian Hamilton operator (which is equivalent to (11) but with real energies ε_k and real non-vanishing interaction $u \equiv \omega$) since $Z^2 + 1 \neq 0$ in this case. In the function space of discrete states, avoided level crossings appear at certain critical values of the parameter considered. These points are usually called diabolic points. The Berry phase characterizes the topological structure of a diabolic point.

3.2. Geometric phase in the extended function space

The geometric phase of eigenvectors of non-Hermitian complex symmetric operators has been considered recently in different papers for paths in parameter space that encircle an

exceptional point. As a result, the geometric phase differs from the Berry phase. This difference is not caused by the fact that the system might pass through an exceptional point by varying parameters while it is impossible to pass the diabolic point. The singular point is encircled in a certain distance, in any case.

In [6], Gamow states are considered. In these studies, an additional part to the Berry phase arises which vanishes with vanishing coupling to the continuum, i.e. when the Gamow states pass into discrete states. This result is related to the fact that, in these studies, the interaction of the Gamow states via a common continuum is not contained. In other papers [13, 16, 17, 30, 36], the interaction of the resonance states via a common continuum is taken into account. Due to this interaction, the resulting geometric phase differs from that obtained in [6], and the limiting case of vanishing coupling to the continuum is not trivial. As a result, a cycle around the exceptional point has to be passed four times in order to produce one full 2π circle in the geometric phase. That means, the exceptional point has to be encircled two times more than a diabolic point in order to restore the wavefunction including its phase.

Supposing that for an N-level system the influence of the other $N-2$ levels onto the two crossing ones is sufficiently weak at the exceptional point, the setup for studying the geometric phase can be modeled by an effective complex symmetric non-Hermitian (2×2) matrix Hamiltonian

$$\mathcal{H}=\begin{pmatrix} e_1 & \omega \\ \omega & e_2 \end{pmatrix}, \qquad \mathcal{H}=\mathcal{H}^T. \tag{10}$$

The complex energies $e_{1,2}$ and the complex channel coupling ω are, in general, parameter dependent. The eigenvalues $\varepsilon_\pm$ and eigenfunctions $\Phi_\pm$ of $\mathcal{H}$ are [30]

$$\varepsilon_\pm=\varepsilon_0\pm\omega\sqrt{Z^2+1} \tag{11}$$

and

$$\Phi_\pm=\begin{pmatrix} 1 \\ -Z\pm\sqrt{Z^2+1} \end{pmatrix} c_\pm, \tag{12}$$

where

$$\varepsilon_0=\frac{1}{2}(e_1+e_2), \qquad Z=\frac{e_1-e_2}{2\omega} \tag{13}$$

and $c_\pm\neq 0$ is complex. The eigenfunctions are biorthogonal according to (4) with (5) and (6). According to (11), it is $Z=\pm\mathrm{i}$ at the exceptional point and, consequently [30, 31],

$$\Phi_\pm=\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} c_\pm \qquad \text{or} \qquad \Phi_\pm=\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} c_\pm \tag{14}$$

with $|c_\pm|\to\infty$. Further [30] $c_+/c_-\to\pm\mathrm{i}$ and $\Phi_+/\Phi_-=\pm\mathrm{i}$ (in agreement with (8)).

In order to receive the geometric phase, the cycle around the exceptional point can be parameterized by $r\,\mathrm{e}^{\mathrm{i}\alpha}$ with $0\leqslant\alpha\leqslant 2\pi$ and $0<r\ll 1$. Then, the evolution along a cycle is given by the transformation matrix $W(\alpha)=\Phi(\alpha)/\Phi(0)$. It follows [30]

$$W(\alpha)\Phi(0)=\mathrm{e}^{-\mathrm{i}\alpha/4}\Phi(0). \tag{15}$$

This result shows the typical fourfold winding around the exceptional point. For illustration, this result can be represented in the following manner according to (8):

$$\begin{aligned}
&1.\quad \text{cycle}: && \varepsilon_\pm\to\varepsilon_\mp && \Phi_\pm\to\pm\mathrm{i}\Phi_\mp \\
&2.\quad \text{cycle}: && \varepsilon_\mp\to\varepsilon_\pm && \pm\mathrm{i}\Phi_\mp\to-\Phi_\pm \\
&3.\quad \text{cycle}: && \varepsilon_\pm\to\varepsilon_\mp && -\Phi_\pm\to\mp\mathrm{i}\Phi_\mp \\
&4.\quad \text{cycle}: && \varepsilon_\mp\to\varepsilon_\pm && \mp\mathrm{i}\Phi_\mp\to\Phi_\pm.
\end{aligned} \tag{16}$$

The geometric phase of the exceptional point is half of the geometric phase (Berry phase) of a diabolic point. In the last case, the Hamiltonian H_B of the system is of standard type for discrete quantum mechanical states. It contains only the internal interaction u. In the first case, however, the Hamiltonian is H_{eff} which contains, additionally to H_B, a second-order term arising from the coupling to the continuum (external interaction v). At the exceptional point, this second-order term becomes the leading term. The difference between the geometric phases in the two cases with H_B and H_{eff}, respectively, illustrates the importance of the interaction via the continuum when the quantum system is open.

3.3. Measurement of geometric phases using microwave resonators

More than ten years ago, the geometric phases of real wavefunctions in nonintegrable quantum billiards are measured by using microwave resonators [22]. The results showed the Berry phase. The cyclic excursion around the diabolic point is achieved by means of parameters that control the shape of the resonator. The geometric phases appear as a sign change of the wavefunction after one cycle. According to expectations it builds up whenever a double degeneracy is encompassed. However, also triple degeneracies lead to a sign change. This last observation caused theoretical studies aiming to explain the data, e.g. [37]. Obviously, an additional mirror symmetry comes into play in triple degeneracies.

Also the topological structure of an exceptional point has been studied in a microwave cavity experiment [19, 20, 38]. To get access to and encircle an exceptional point in the experiment, an absorptive system is used. It consists of two semicircular cavities of slightly different sizes which can be coupled by adjusting the opening of a slit between them. The second parameter is given by the distance between the centers of the cavity and a teflon semicircle placed on one side of the cavity. In the experiment, the real and imaginary parts of the eigenvalues and eigenvectors of the non-Hermitian Hamilton operator H_{eff} are traced on a closed path around the exceptional point. Along this path the complex eigenvalues are interchanged, but their trajectories never cross each other (i.e. avoid crossing in the complex energy plane). The experimental results [19] confirm the expectations: a cycle around the exceptional point in parameter space has to be passed four times in order to produce one full 2π cycle in the geometric phase.

In a next experiment, the authors studied the phase difference between the two eigenvectors in approaching the exceptional point [38]. As a result, the phase difference between the two modes changes from π at large distance between them to $\pi/2$ in approaching the exceptional point. This result has been explained in [38] by the assumption that the state at the exceptional point is a chiral state.

The experimental results [38] can be explained also by means of the phase rigidity r_λ of the complex eigenfunctions ϕ_λ of the non-Hermitian Hamilton operator H_{eff} [33]. The phase rigidity drops smoothly from its maximum value $r_\pm = 1$ far from the exceptional point (with the phase difference π (or 2π) between the wavefunctions of isolated resonance states) to its minimum value $r_\pm = 0$ at the exceptional point (with the phase difference $\pm\pi/2$ according to (8)). This interpretation explains, in a natural manner, the experimentally observed smooth reduction of the phase difference in a comparably large parameter range. Further, the phase jump by $\pi/4$ occurring in passing the exceptional point [13, 30], is directly related to the fact that this point has to be encircled four times in order to restore the wavefunction including its phase. It corresponds therefore to the geometric phase of this point that is measured in [19]. In this manner, the experimental results can be considered to demonstrate the (parametric) dynamics of open quantum systems which is generated by the interaction of resonance states via the continuum.

4. Phase transition in an open quantum system

4.1. Schematic model

Let us consider first a simple model with the following restrictions. The effective Hamilton operator is approximated by

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}^0 - \mathrm{i}\alpha V V^+, \tag{17}$$

where VV^+ is a Hermitian operator. The first term $\tilde{\mathcal{H}}^0$ describes the internal structure of the unperturbed system in the Q subspace while the second term describes the coupling between the two subspaces in a simplified manner with the parameter α characterizing the mean coupling strength between discrete and continuum states. Further restrictions are the following: (i) $\tilde{\mathcal{H}}^0$ is supposed to be diagonal, i.e. $\tilde{\mathcal{H}}$ is written in the eigenbasis of $\tilde{\mathcal{H}}^0$, (ii) the number M of resonance states is large, (iii) the number K of open decay channels is small, and (iv) the energy dependence of the eigenvalues and eigenfunctions of the effective Hamiltonian is weak, in spite of the large number M of states, and therefore neglected. The rank of $\tilde{\mathcal{H}}^0$ is equal to the number M of states considered. The coupling matrix V is a $K \times M$ matrix and the matrix element V_k^c describes the coupling of the discrete state k to the channel c; $k = 1, \ldots, M$; $c = 1, \ldots, K$. Thus, the rank of VV^+ is K.

Let us first consider the case with real α. If $|\alpha|$ is small, the second term in (17) can be regarded as a small perturbation. In this case, $\tilde{\mathcal{H}}$ has M almost real eigenvalues. If, however, $|\alpha| \gg 1$, the first term $\tilde{\mathcal{H}}^0$ appears as a small perturbation and the matrix VV^+ provides K eigenvalues. Therefore, $\tilde{\mathcal{H}}$ has K eigenvalues with large imaginary part. In between these two limiting cases, a transition occurs between both regimes. Crucial for this transition is the distribution of the exceptional points in the complex energy plane which is exclusively fixed by the distribution of the matrix elements of $\tilde{\mathcal{H}}^0$ and VV^+.

In [17, 21], the model (17) has been used in order to investigate if and under which conditions this transition can be understood as a phase transition. The study is performed with $K = 1$ meaning that finally one short-lived mode is formed after $M - 1$ avoided or true crossings with $M - 1$ resonance states. Thus, $M - 1$ exceptional points are expected to appear. It is shown analytically [21] that, in the limit $M \to \infty$, a simultaneous coalescence of all eigenvalues occurs at a finite real value of α, if the distribution of the real eigenvalues $\tilde{E}_k$ of $\tilde{\mathcal{H}}^0$ and the coupling matrix elements v_k (i.e. the elements of the vector V) are appropriately chosen. In that case, all $M - 1$ exceptional points accumulate at one single point in the complex parameter plane. The most illustrative case is a picket-fence model with equal distance between the states and equal coupling strength of all the states to the continuum, $v \equiv v_k$ for all k. More generally, an appropriate condition can be achieved when regions with a smaller level density of the unperturbed states are stronger coupled to the decay channel than those with a higher level density. For example for the level distribution $\tilde{E}_k^2 \approx x^t$ and the coupling strength $v_k^2 \approx x^r$, such a situation appears [21] when $2(r + 1) = t$. Here, $\alpha_{\mathrm{cr}} = (r + 1)/\pi = t/(2\pi)$. For the picket-fence model, it is $t = 2$, $r = 0$ and $\alpha_{\mathrm{cr}} = 1/\pi$.

If $2(r + 1) > t$, $\alpha_{\mathrm{cr}} \to 0$ in the limit $M \to \infty$. That means, there exists a state with large decay width at any finite value $\alpha > 0$. If however $2(r + 1) < t$, it follows $\alpha_{\mathrm{cr}} \to \infty$, i.e. the reorganization process occurs always locally and does not finish for any finite arbitrary high value α. In this case also an eigenvalue with large imaginary part appears, but now via a successive but infinite chain of level repulsions.

Although mathematically the limit $M \to \infty$ is required for the simultaneous coalescence of all eigenvalues, the evolution of the system traced by varying α along the real axis resembles nicely all features of a second-order phase transition even for $M = 10^2$ states (when $2(r + 1) = t$) [21]. Here the coupling strength α acts as a control parameter while the

imaginary part of the large eigenvalue plays the role of an order parameter [21]. Furthermore, it could be shown that the relation between the distribution of unperturbed states and the coupling strength, i.e. between r and t, has to be fulfilled only approximately. If either the level density of $\tilde{\mathcal{H}}^0$ or the coupling matrix VV^+ (or both) are additionally altered by noisy perturbations, an abrupt transition occurs at $\alpha_{\rm cr}$ numerically even when only a comparably small configuration space is considered [21]. Under the condition $2(r+1) \approx t$, all exceptional points of the system accumulate at some finite real value of the parameter $\alpha = \alpha_{\rm cr}$ (see figure 2 of [17]). In the limit $M \to \infty$ a perfect coalescence of an infinite number of exceptional points is succeeded.

It is interesting to remark that, in the case of a phase transition, the short-lived eigenstate λ_0 is collective in the sense that the number of principal components of its eigenfunction jumps abruptly to its maximal value at the critical value $\alpha_{\rm cr}$, i.e. its wavefunction consists of a (constructive) superposition of all eigenstates of $\tilde{\mathcal{H}}^0$. The wavefunctions of all the other $M-1$ eigenstates of $\tilde{\mathcal{H}}$, however, stay almost pure in this basis. In this sense, the short-lived eigenmode with a large imaginary part $\Gamma_0/2$ of its eigenvalue is an extremely collective state. This is true, although Γ_0 is much smaller than the extension of the spectrum at $\alpha = \alpha_{\rm cr}$.

As a conclusion, the results obtained analytically and numerically on the basis of the schematic non-Hermitian Hamilton operator $\tilde{\mathcal{H}}$, equation (17), with real values of the parameter α showed the appearance of a second-order phase transition[3]. In the transition from the low coupling regime to the high one a collective mode occurs suddenly in one point of the parameter space. This point is the accumulation point of exceptional points in the function space considered.

4.2. Realistic models

A more realistic situation is obtained when the coupling parameter is chosen to be complex, i.e. $\alpha \to \alpha\, {\rm e}^{{\rm i}\varphi}$ is considered in (17). In this case, the system can no longer evolve through the accumulation point of the exceptional points [21]. The reason is that the accumulation point of the exceptional points is at the real α-axis. The system can therefore not hit the accumulation point, but has to pass it at a certain distance in the complex parameter space. Hence, the reorganization process is washed out. That means, a critical region of reorganization of the system can be observed as a function of α, but a strict phase transition cannot occur when α is complex [21].

Such a situation is studied numerically in a realistic model for an open microwave cavity with the full Hamiltonian $H_{\rm eff}$, equation (2) [15]. Here, the transmission through quantum dots of different shape is calculated as a function of the coupling strength to the attached leads. In these calculations, the number of states in the considered energy region is finite and the transition between two different scenarios ('phase transition') is spread over a comparably large range of the parameter varied. It occurs from a scenario with isolated resonances to another one with narrow resonances superposed by a background. In the transition region, the density of exceptional points is high and the resulting resonance trapping phenomenon occurs hierarchically in a certain finite parameter range.

The most interesting result of these calculations is that the transmission through the system is enhanced in the whole parameter range with densely lying exceptional points. The enhancement is a consequence of the fact that the phase rigidity of the eigenfunctions of $H_{\rm eff}$ is reduced in the regime with overlapping resonances [15, 33]. Of special interest is the appearance of whispering gallery modes along the convex boundary of a cavity.

[3] Our considerations hold at zero temperature.

As a conclusion, the physically interesting phenomenon of enhanced transmission through the system in a certain finite parameter interval is a consequence of the interplay between the Hermitian and non-Hermitian parts of $H_{\rm eff}$. It prevents an accumulation of the exceptional points in one point, and the transition between the different scenarios takes place hierarchically in a finite range of the parameter value. With the approximation

$$\mathrm{Re}(H_{\rm eff} - H_B) = \mathrm{Re}\left(\sum_C V_{BC}\frac{1}{E^+ - H_C}V_{CB}\right) = 0 \tag{18}$$

and in the limit of large N, the parameter range with enhanced transmission shrinks to one point as discussed in section 4.1.

5. Examples studied numerically in the framework of realistic models

5.1. Level repulsion and widths bifurcation

According to the eigenvalue equation (11), level repulsion is related to $\mathrm{Re}(\omega\sqrt{Z^2+1})$ while widths bifurcation is determined by $\mathrm{Im}(\omega\sqrt{Z^2+1})$. The exceptional point is well defined by $Z^2+1=0$, i.e. by $Z=\pm\mathrm{i}$. By varying only one parameter, this point in the continuum is seldom passed. Mostly the levels avoid crossing in the complex energy plane at some value of the control parameter α. The regime of level repulsion and that of widths bifurcation therefore do not finish and start, respectively, at the same point of α. Nevertheless, regions with predominant level repulsion are well separated in parameter space from those with predominant widths bifurcation (and level clustering). For some illustrative examples see [11].

Numerical calculations for different realistic models in the framework of the FPO method have supported this general picture, see [8, 9] for atoms in a laser field and for quantum billiards with one [39] or two attached leads [12, 13, 15]. The widths bifurcation has been seen even experimentally in tracing the eigenvalue trajectories in a quantum billiard as a function of the coupling strength between cavity and attached lead [27]. Similar results are obtained in many other calculations performed in the framework of other models. As an example, we refer here only to the results presented in [6].

Level repulsion has been discussed in many papers for many years, especially in relation to quantum chaos. Since it is related to $\mathrm{Re}(\omega\sqrt{Z^2+1})$ (according to the eigenvalue equation (11)) it appears also in closed systems. For details see, e.g., the textbooks [40, 41].

The scenario in open quantum systems is much richer. Here, the effects arising from $\mathrm{Im}(\omega\sqrt{Z^2+1}) \neq 0$ appear, i.e. from the bifurcation of the decay widths traced as a function of a certain control parameter α. An interesting phenomenon that is directly related to avoided level crossings and is of importance for applications, is high-order harmonic generation occurring in driven two-level atoms [42]. Another phenomenon is the appearance of bound states in the continuum, i.e. of states whose widths vanish although their position in energy is above the decay threshold and their decay is not forbidden by any selection rule. The relation of this phenomenon to avoided level crossings in the continuum has been first discussed in [43] for atoms. Due to the interplay between the Hermitian and non-Hermitian parts of the Hamilton operator $H_{\rm eff}$ in realistic models, bound states in the continuum occur also for realistic (finite) parameter values [8, 9]. In atomic physics, this phenomenon is known as population trapping studied first in a time-dependent formalism [44]. The bound states in the continuum appear, in this description, as resonance states the population probability of which is time independent.

Bound states in the continuum have been found and related to the avoided level crossing phenomenon also in open quantum dots [12, 14, 45]. Here, they cause zeros in the transmission probability. More detailed results can be found in [46].

5.2. Phase rigidity of the wavefunctions and enhancement of observables

The eigenfunctions of $H_{\rm eff}$ in the vicinity of exceptional points are much less studied than the eigenvalues. From (12) follows (8) at the exceptional point as shown in, e.g., [30, 32]. In other approaches, e.g. [17, 19, 20, 38], the relation $\phi_k \to \phi_l; \phi_l \to -\phi_k$ is used instead of $\phi_k \to \pm i\phi_l$, equation (8).

In recent studies, the phase rigidity of the eigenfunctions of $H_{\rm eff}$ is shown to be reduced in approaching an exceptional point [33]. This result agrees with experimental results [38] obtained in approaching an exceptional point in a microwave cavity (section 3.3).

An interesting effect caused by the reduced phase rigidity of the wavefunctions in the regime of overlapping resonances is the enhancement of observable values as, e.g., of the transmission probability through a quantum dot [15]. An enhancement occurs by hierarchical resonance trapping at true and avoided level crossings in the complex energy plane. It is directly related to the fact that the system passes through a region with a high density of exceptional points [33]. Finally, two short-lived collective states are formed which are aligned each to one of the two channel wavefunctions. As a function of a parameter that characterizes the degree of opening of the system, the transmission occurs via N single resonances in the scenario at low opening and via $N-2$ narrow resonances superposed by a background in the scenario at strong opening. The transition between the two scenarios takes place in the regime in which the resonance states strongly overlap and the rigidity of their phases is reduced [15].

The best illustrative case is a cavity with a convex boundary where whispering gallery modes appear [15, 47]. The transmission through these modes is enhanced. It occurs in the short-time scale as a shot-noise analysis has shown [48]. The transmission in the regime of overlapping resonances does not show single resonance peaks but is plateau-like [15, 47] being a typical feature of the reduced phase rigidity of the wavefunctions.

5.3. Topological structure of exceptional points

In analytical and numerical calculations for quantum dots, the topological structure of exceptional and diabolic points is studied [13]. The results show that both types of singular points differ substantially from one another: (i) the encircling of a diabolic point gives rise to a geometric phase in the closed system (Berry phase), and does not cause any phase in the open system, (ii) the encircling of the exceptional point gives rise to a geometric phase in the open system but has no effect in the closed system. This result follows from the different Hamilton operators in the two cases. In the first case, $H_{\rm eff}$ (equation (2)) reduces to H_B being the standard Hermitian Hamilton operator of the closed system. In the second case, however, the most important part of $H_{\rm eff}$ is the non-Hermitian interaction part via the continuum being formally a second-order term. The corresponding phase is half of the Berry phase.

The results obtained for the exceptional points in quantum dots agree with those obtained analytically [30] for the general case. They agree also with the results obtained experimentally on microwave resonators [19, 20].

5.4. Crossing points of three and more eigenvalues

The wavefunctions at crossing points of the eigenvalues of three resonance states are studied for a quantum dot with a small number of resonance states [13]. In this special case, the

triple exceptional point turns to a double exceptional point since the third eigenvalue remains unaffected by the crossing. This behavior is caused, surely, by some symmetry involved in the system. It is not in contradiction with the general conclusion that the real part of the non-Hermitian part of H_{eff} prevents the system to cross the accumulation point of exceptional points (section 4.2).

More theoretical and experimental studies have to be performed in order to find an answer whether or not non-contractible multiple exceptional points exist in realistic systems.

6. Conclusions

In this paper, we reviewed the role exceptional points play in the dynamics of open quantum systems. The spectroscopic properties of an open quantum system are well described by the eigenvalues and eigenfunctions of the non-Hermitian Hamilton operator H_{eff} (equation (2)). The eigenvalue trajectories traced in the complex plane as a function of a parameter, may cross or avoid crossing in the complex energy plane. The geometric phase of an exceptional point (or an avoided level crossing) differs from the Berry phase. The reason for this difference is the different function space basic for the description of the two different situations (open and closed quantum systems, respectively). It may be considered as a signature of the additional correlations induced in an open quantum system by the coupling to the common continuum. In physical systems, the exceptional points separate the scenario with level repulsion from that with widths bifurcation (and level clustering). Widths bifurcation may lead to the interesting phenomenon of population trapping, i.e. to the appearance of bound states in the continuum (meaning some stabilization of the system at certain parameter values).

In this paper, we concentrated on the wavefunctions of the system being eigenfunctions of the non-Hermitian Hamilton operator H_{eff}, equation (2). They contain correlations caused by the standard internal interaction as well as by the external interaction of the resonance states via the common continuum. The phases of the eigenfunctions of H_{eff} depend on the degree of overlapping of resonances. The phase rigidity is reduced in the vicinity of an exceptional point and allows, in this manner, an alignment of a few wavefunctions of the system each with one of the (channel) wavefunctions of the environment. This process is basic for the dynamics of open quantum systems.

Although exceptional points are single points in the complex plane, they determine decisively the dynamics of open quantum systems. In spite of many studies, some problems are still unsolved.

(i) The experimental data received in approaching the exceptional point in a microwave cavity are interpreted by the authors as observation of a chiral state [38] in spite of the fact that the exceptional point is a single point in the continuum. An alternative explanation of the data considers the reduction of the phase rigidity in approaching an exceptional point (section 3.3). The advantage of the last interpretation is the fact that the phase rigidity is reduced not only at the exceptional point. The reduction can be traced in a large parameter range in approaching the exceptional point (in agreement with the experimental results). It would be interesting to also look for the phase jump appearing at the exceptional point (section 2.3).

(ii) The transmission through a microwave cavity is enhanced in approaching the critical value of the coupling strength between system and attached leads, i.e. the critical value of the degree of opening of the system. Here, the transition from the transmission scenario via single resonance states to that via resonance states superposed by a background takes place in a region with high density of exceptional points (section 4.2). In this region, the

phase rigidity of the wavefunctions is reduced. Finally, a new decay channel opens. The relation between such a 'phase transition' and the position of a threshold for opening a new decay channel needs further investigation.

(iii) The question whether or not true multiple exceptional points exist is unclear. Theoretical studies on a realistic system do not support a true accumulation of exceptional points (section 4.2). However, they do not clearly exclude an accumulation of a few of them, i.e. the existence of true multiple exceptional points. Further studies of triple exceptional points will surely give a contribution to the solution of this problem. Additionally, they will throw light on the relation to the Berry phase and the appearance of internal symmetries (section 3.3).

A solution of these problems will give invaluable contributions to a better understanding of the dynamics of open quantum systems.

Acknowledgments

Valuable discussions with M Berry are gratefully acknowledged. We thank the Max Planck Institute for the Physics of Complex Systems for hospitality.

References

[1] Kato T 1966 *Perturbation Theory for Linear Operators* (Berlin: Springer)
[2] Baumgärtel H 1984 *Analytic Perturbation Theory for Matrices and Operators* (Berlin: Akademie-Verlag)
Baumgärtel H 1985 *Operator Theory: Adv. Appl.* 15 (Basel: Birkhäuser)
[3] Feshbach H 1958 *Ann. Phys., NY* **5** 357
Feshbach H 1962 *Ann. Phys., NY* **19** 287
[4] Rotter I 2007 *Preprint* 0711.2926
[5] Newton R G 1982 *Scattering Theory of Waves and Particles* (New York: Springer)
[6] Mondragon A and Hernandez E 1993 *J. Phys. A: Math. Gen.* **26** 5595
Hernandez E and Mondragon A 1994 *Phys. Lett.* B **326** 1
Hernandez E, Jauregui A and Mondragon A 2003 *Phys. Rev.* A **67** 022721
Hernandez E, Jauregui A and Mondragon A 2005 *Phys. Rev.* E **72** 026221
[7] Vanroose W, VanLeuven P, Arickx F and Broeckhove J 1997 *J. Phys. A: Math. Gen.* **30** 5543
Vanroose W 2001 *Phys. Rev.* A **64** 062708
[8] Magunov A I, Rotter I and Strakhova S I 1999 *J. Phys. B: At. Mol. Opt. Phys.* **32** 1669
[9] Magunov A I, Rotter I and Strakhova S I 2001 *J. Phys. B: At. Mol. Opt. Phys.* **34** 29
[10] Solov'ev E A 1981 *Sov. Phys.—JETP* **54** 893
Solov'ev E A 1989 *Sov. Phys. Usp.* **32** 228
Briggs J S, Savichev V I and Solov'ev E A 2000 *J. Phys. B: At. Mol. Opt. Phys.* **33** 3363
Solov'ev E A 2005 *J. Phys. B: At. Mol. Opt. Phys.* **38** R153
[11] Rotter I 1991 *Rep. Prog. Phys.* **54** 635
Okolowicz J, Ploszajczak M and Rotter I 2003 *Phys. Rep.* **374** 271
[12] Rotter I and Sadreev A F 2004 *Phys. Rev.* E **69** 066201
[13] Rotter I and Sadreev A F 2005 *Phys. Rev.* E **71** 036227
[14] Rotter I and Sadreev A F 2005 *Phys. Rev.* E **71** 046204
[15] Bulgakov E N, Rotter I and Sadreev A F 2006 *Phys. Rev.* E **74** 056204
Bulgakov E N, Rotter I and Sadreev A F 2007 *Phys. Rev.* B **76** 214302
[16] Cartarius H, Main J and Wunner G 2007 *Phys. Rev. Lett.* **99** 173003
[17] Heiss W D, Müller M and Rotter I 1998 *Phys. Rev.* E **58** 2894
[18] Heiss W D 1999 *Eur. Phys. J.* D **7** 1
[19] Dembowski C, Gräf H D, Harney H L, Heine A, Heiss W D, Rehfeld H and Richter A 2001 *Phys. Rev. Lett.* **86** 787
[20] Dembowski C, Dietz B, Gräf H D, Harney H L, Heine A, Heiss W D and Richter A 2004 *Phys. Rev.* E **69** 056216
[21] Jung C, Müller M and Rotter I 1999 *Phys. Rev.* E **60** 114
[22] Lauber H M, Weidenhammer P and Dubbers D 1994 *Phys. Rev. Lett.* **72** 1004

[23] Graefe E M 2007 A non-Hermitian two-mode Bose-Hubbard system *Talk at the 6th International Workshop on Pseudo Hermitian Hamiltonians in Quantum Physics (London, 16–18 July 2007)*
[24] Znojil M 2008 *J. Phys. A: Math. Theor.* **41** 244027
[25] Heiss W D 2008 *J. Phys. A: Math. Theor.* **41** 244010
[26] Sinha A 2007 A class of Non Hermitian Models with real energies *Talk at the 6th International Workshop on Pseudo Hermitian Hamiltonians in Quantum Physics (London, 16–18 July 2007)*
Sinha A 2007 Symmetry breaking at exceptional points *Talk at the 5th International Symposium on Quantum Theory and Symmetries (Valladolid, 22–28 July 2007)*
[27] Persson E, Rotter I, Stöckmann H J and Barth M 2000 *Phys. Rev. Lett.* **85** 2478
Stöckmann H J, Persson E, Kim Y H, Barth M, Kuhl U and Rotter I *Phys. Rev.* E **65** 066211
[28] Rotter I 2003 *Phys. Rev.* E **68** 016211
[29] Rotter I 2004 *Acta Phys. Pol.* B **35** 1269
[30] Günther U, Rotter I and Samsonov B F 2007 *J. Phys. A: Math. Theor.* **40** 8815
[31] Müller M, Dittes F M, Iskra W and Rotter I 1995 *Phys. Rev.* E **52** 5961
[32] Rotter I 2001 *Phys. Rev.* E **64** 036213
[33] Rotter I 2007 *J. Phys. A: Math. Theor.* **40** 14515
[34] Berry M V 1984 *Proc. R. Soc. Lond.* A **392** 45
[35] Berry M V and Wilkinson M 1984 *Proc. R. Soc. Lond.* A **392** 15
[36] Keck F, Korsch H J and Mossmann S 2003 *J. Phys. A: Math. Gen.* **36** 2125
[37] Pistolesi F and Manini N 2000 *Phys. Rev. Lett.* **85** 1585
Manini N and Pistolesi F 2000 *Phys. Rev. Lett.* **85** 3067
Samuel J and Dhar A 2001 *Phys. Rev. Lett.* **87** 260401
Samuel J and Dhar A 2002 *Phys. Rev.* A **66** 044102
[38] Dembowski C, Dietz B, Gräf H D, Harney H L, Heine A, Heiss W D and Richter A 2003 *Phys. Rev. Lett.* **90** 034101
[39] Persson E, Pichugin K, Rotter I and Seba P 1998 *Phys. Rev.* E **58** 8001
Seba P, Rotter I, Müller M, Persson E and Pichugin K 2000 *Phys. Rev.* E **61** 66
Rotter I, Persson E, Pichugin K and Seba P 2000 *Phys. Rev.* E **62** 450
[40] Haake F 2004 *Quantum Signatures of Chaos* (Berlin: Springer)
[41] Stöckmann H J 2000 *Quantum Chaos* (Cambridge: Cambridge University Press)
[42] Faria C F M and Rotter I 2002 *Phys. Rev.* A **66** 013402
[43] Friedrich H and Wintgen D 1985 *Phys. Rev.* A **32** 3231 and 3964
[44] Kylstra N J and Joachain C J 1998 *Phys. Rev.* A **57** 412
[45] Bulgakov E N, Rotter I and Sadreev A F 2007 *Phys. Rev.* A **75** 067401
[46] Sadreev A F, Bulgakov E N and Rotter I 2005 *JETP Lett.* **82** 498
Sadreev A F, Bulgakov E N and Rotter I 2006 *Phys. Rev.* B **73** 235342
Bulgakov E N, Pichugin K N, Sadreev A F and Rotter I 2006 *JETP Lett.* **84** 430
[47] Nazmitdinov R G, Pichugin K N, Rotter I and Seba P 2001 *Phys. Rev.* E **64** 056214
Nazmitdinov R G, Pichugin K N, Rotter I and Seba P 2002 *Phys. Rev.* B **66** 085322
[48] Nazmitdinov R G, Sim H S, Schomerus H and Rotter I 2002 *Phys. Rev.* B **66** 241302